TOWARDS A DESCRIPTIVE METAPHYSICS OF MATHEMATICS

Describing Mathematics in Typed Predicate Logic

Typed Predicate Logic

- Typed Predicate Logic was developed as a framework for the articulation and study of the rich variety of logical systems.
- It is a system of logic in which the notion of *type theory* plays a parallel role to that of a first-order theory in first-order logic.
- Standard logics are formulated as type theories.

Background

- Simple and Ramified type theories (Russell)
- Intensional simple type theory (Montague)
- Intensional ramified type theories (Thomason, Turner)
- Logics of Propositions, Properties and Truth (Aczel, Turner)
- Constructive type theories (Curry, Howard, Martin-Löf)
- Theories of Operations and Types (Feferman and Turner)
- Second Order Lambda Calculus and the Theory of Constructions (Reynolds, Girard, Huet, Coquand)
- Combinatorial Logic (Curry)

Areas of Application

1) **Logics of Computation**: Hoare logic, Dynamic Logic, Domain logic, Process algebra,...
2) **Formal Ontology**: sortal logics, mereology, events and time, process logic etc.
3) **Natural Language Semantics**: Montague grammar, Situation theory, Property theories, Discourse Representation Theory..
4) **Mathematical Ontology**: descriptive metaphysics for mathematics.

We shall concentrate on giving an exposition of TPL via its application to the last area. We shall mention the others.

OUTLINE

I. Ontological Reduction in The Foundations of Mathematics
II. Mathematical Practice
III. Typed Predicate Logic
IV. Type Theories
V. Definitions
VI. Subtypes
VII. Abstract types
VIII. Polymorphism in Mathematics
IX. Sorts and Types
X. Other Applications

I. ONTOLOGICAL REDUCTION IN THE FOUNDATIONS OF MATHEMATICS

The Foundations of Mathematics

- Many of the foundational schools have sought to reduce mathematics to a few ontological notions.
- The motivation was largely epistemological i.e., to ensure the soundness of mathematical knowledge.

Logicism

- The program of (Frege, Russell and Whitehead) sought to reduce mathematics to *logic*.
- This concerned not only the proofs of mathematics but also the objects themselves; they were taken to be *logical* notions.
- Frege’s system turned out to be inconsistent. Russell introduced types to avoid the paradoxes: simple type theory to avoid the logical paradoxes and ramified type theory to avoid the semantic ones.

Hilbert’s Program

- Hilbert’s program sought to reduce (infinitistic) mathematics to a base of finite notions.
- Again, not for ontological ends but in order to show the consistency of the whole of mathematics.
- Hilbert’s program was taken to be overthrown by the incompleteness theorems of Gödel: any sufficiently expressive system will have (true) sentences that are not decidable by the system. Indeed, the statement of consistency is not decidable.
- Results of Paris and Harrington exhibit a sentence in actual number theory – The strengthened finite Ramsey theorem.
- More recent results show that ZF+LCA decides the theorem. We have come full circle from Hilbert’s program in that infinitistic notions decide finite ones.

Constructivism

- Constructivism restricts the logic (LEM) and the class of mathematical objects. Modern justification is given in terms of the interpretation of the logical connectives (Dummett).
- Proof conditions rather than truth conditions. Objects must be constructible.
- Taken to be consistent (but only consistent) with Church-Turing thesis i.e., constructions are Turing computable.
- IZF and the link between logic and ontology

Set and Theoretic Foundations

- Set theoretic foundations – a single ontology of sets. All mathematical objects, including number, are represented as sets.
- Standard type theories also reductionist - they fix the class of types.
- Category theory

Predicativism

- Emanates from Russell's type theory
- Impredicative definitions banned- the class of all classes.
- Ontological restrictions for epistemological reasons.
- Feferman developed analysis with predicative definitions.

Scientific Naturalism

- Quine, Putnam and Scientific naturalism about mathematics.
- An ontological commitment to some mathematical entities is inherent in our best scientific theories.
- Indispensability arguments.
- Quine accepted only a subsystem of set theory.

Fictionalism

- A form of Nominalism
- An explicitly reductionist program.
- Hartry Field: attempt to undermine the indispensability arguments.
- The mathematics used in science is a conservative extension of a nominalistic language/theory.
- Not dissimilar to Hilbert but with physical objects replacing finite ones.

What Numbers Cannot be

- Benacerraf argument concerns Frege’s representation of numbers as objects.
- Set theoretic reduction: any representation of numbers gives them extraneous properties.
- E.g. 3 is a member of 4.
- Not faithful.

All Reductionist

- There is no intention to describe (classical) mathematical activity as it is.
- Ontological Reduction
- Epistemological motivation

II. MATHEMATICAL PRACTICE

Descriptive Metaphysics of Mathematical Practice

- Consistent with the *new wave* in the philosophy of mathematics that emphasises mathematical practice.
- No reduction for epistemological or nominalistic purposes.
- The underlying ontology is reflected directly in the logical systems.
- Of course, there is always some reduction that occurs inside the system via definitions.
- In any branch of mathematics, the decision concerning which notions to take as primitive and which are to be defined requires mathematical judgement.

Faithful and Adequate Formalisation

Existing formalisations are not always

- *Adequate*: all concepts of the informal theory/system are covered
- *Faithful*: does not go beyond the informal theory

The representation of many mathematical notions is rarely faithful. Feferman.

Mathematical Naturalism

- Consistent with the mathematical naturalism of Maddy.
- Mathematics taken at face value. It has its own internal notions of legitimacy. E.g. *maxify* and *unify* in set theory to reject the constructible universe (V=L). ZF+LCA will be accepted or not on mathematical grounds.
- However, whether we give ZF+LCA a realist or an antirealist interpretation is a philosophical question.
- Scientific naturalism about mathematics is rejected. However, she adopts a form of scientific naturalism about the philosophy of mathematics. Such questions are to be determined by scientific standards of simplicity.
- Not clearly a clean separation: might not scientific simplicity argue for a constructive interpretation?

III. TYPED PREDICATE LOGIC

Typed Predicate Logic as an Ontological Framework

- Typed Predicate Logic is a flexible logical framework in which a rich variety of ontological systems may be directly articulated.
- Different *types* of abstract objects.

Types and Grammar

- Type Theory extends syntax: the need for a flexible theory of syntax that extends the traditional approach via context free grammar.
- Grammar and logic intertwined
- Gives flexibility for the formulation of a rich variety of theories without standard syntax getting in the way.

Judgements of TPL

T *type*

$t{:}T$

Φ *prop*

Φ *(true)*

Contexts and Sequents

1. $c = x_1 : T_1, \ldots, x_n : T_n$ ....*type contexts*

2. $\Gamma = x_1 : T_1, . \varphi_1, \ldots., x_n : T_n, \ldots, \varphi_m$ ....*for general ones*

3. $\Gamma \vDash \varphi$ *Sequents*

Relations, Functions and Types

$$x_1 : T_1, \ldots.., x_n : T_n \ \vdash R_{T_1, \ldots.., T_n}(x_1, \ldots.., x_n) \ \mathit{prop}$$

$$x_1 : T_1, \ldots.., x_n : T_n \ \vdash F_{T_1, \ldots.., T_n}(x_1, \ldots.., x_n) : T[x_1, \ldots.., x_n]$$

$$x_1 : T_1, \ldots.., x_n : T_n \ \vdash O_{T_1, \ldots.., T_n}(x_1, \ldots.., x_n) \ \mathit{type}$$

Grammar Rules

Syntax given by rules. This permits the expression of systems that involve a rich notions of type including dependency and self application

1. *Propositions* closed under the standard connectives

$$\frac{\Gamma \vDash \varphi\ prop \qquad \Gamma \vDash \psi\ prop}{\Gamma \vDash \varphi \wedge \psi\ prop}$$

3. And closed under typed Quantifiers

$$\frac{\Gamma, x{:}T \vDash \varphi\ prop}{\Gamma \vDash \forall x{:}T.\varphi\ prop} \qquad\qquad \frac{\Gamma, x{:}T \vDash \varphi\ prop}{\Gamma \vDash \exists x{:}T.\varphi\ prop}$$

* Note the dependency of the judgement

Logical Rules

$$\frac{\Gamma \vDash \varphi \qquad \Gamma \vDash \psi}{\Gamma \vDash \varphi \wedge \psi}$$

$$\frac{\Gamma \vDash \varphi \qquad \Gamma \vDash \psi \; prop}{\Gamma \vDash \varphi \vee \psi}$$

$$\frac{\Gamma, \varphi \vDash \psi}{\Gamma \vDash \varphi \to \psi}$$

$$\frac{\Gamma \vDash \varphi \qquad \Gamma \vDash \varphi \to \psi}{\Gamma \vDash \psi}$$

Equality

$$\frac{\Gamma \vDash a : T \qquad \Gamma \vDash b : T}{\Gamma \vDash a =_T b \; Prop} \qquad \frac{\Gamma \vDash a =_T b \qquad \Gamma \vDash \phi[a] \qquad \Gamma, x : T \vDash \phi \; Prop}{\Gamma \vDash \phi[b]}$$

$$\frac{\Gamma \vDash a : T}{\Gamma \vDash a =_T a}$$

Coherence

Theorem: *If* $\Gamma \vDash \psi$ *then* $\Gamma \vDash \psi$ *prop*

Proof *By induction on the derivations: rule by rule argument.*●

- *Compare with the older Curry systems of logical anarchy.*

Grammar Independence

Definition

A system is Grammar Independent if

1. $\Gamma \vDash \psi$ *prop then* $c_\Gamma \vDash \psi$ *prop*
2. $\Gamma \vDash T$ *type then* $c_\Gamma \vDash T$ *type*
3. $\Gamma \vDash t{:}T$ *then* $c_\Gamma \vDash t{:}T$

i.e., type conclusions only depends on type contexts.

The full logical system is not decidable.

Question: When is a system a conservative extension of a grammar independent system?

IV. TYPE THEORIES

Numbers

$$\mathbf{N_0} \quad N \ \mathit{type}$$

$$\mathbf{N_1} \quad 0 : N \qquad\qquad \mathbf{N_2} \quad \frac{a : N}{a^+ : N}$$

$$\mathbf{N_3} \quad \frac{\phi[0] \qquad x : N, \phi[x] \vdash \phi[x^+]}{x : N \vdash \phi[x]}$$

$$\mathbf{N_4} \quad \frac{a : N}{a^+ \neq 0} \qquad\qquad \mathbf{N_5} \quad \frac{a^+ =_N b^+}{a =_N b}$$

Cartesian Products

$$\frac{T \text{ type} \qquad S \text{ type}}{T \otimes S \quad \text{type}}$$

$$\frac{t{:}T \qquad s{:}S}{(t, s){:}T \otimes S}$$

$$\frac{t{:}T \otimes S}{lt{:}T}$$

$$\frac{t{:}T \qquad s{:}T}{l(t, s)=t}$$

The Type Lambda Calculus

$$\frac{S\ \mathit{type} \qquad T\ \mathit{type}}{T \Rightarrow S\ \mathit{type}}$$

$$\frac{x{:}T \vDash t{:}S}{\lambda x{:}T.t : T \Rightarrow S} \qquad\qquad \frac{f{:}T \Rightarrow S \qquad t{:}T}{ft{:}S}$$

$$\frac{x{:}T \vDash t{:}S \qquad s{:}T}{(\lambda x{:}T.t)s = t[s/x]}$$

Functional Type Theory

- FTT=Th(N,$\otimes$,$\Rightarrow$)
- Basic types: Numbers
- Constructors: Products, Arrow

Theorem *FTT is a grammar independent theory*

Sets

$$\frac{T\ type}{Set(T)\ type}$$

$$\frac{x : T \vdash \phi\ prop}{\{x : T \cdot \varphi\} : Set(T)} \qquad \frac{s : Set(T) \qquad t : T}{t \in s\ prop}$$

$$\frac{x : T \vdash \phi\ prop \qquad t : T \qquad \phi[t]}{t \in \{x : T \cdot \varphi\}}$$

$$\frac{x : T \vdash \phi\ prop \qquad t \in \{x : T \cdot \phi\}}{\phi[t]}$$

$$\frac{x : T \vdash \phi\ prop \qquad t \in \{x : T \cdot \phi\}}{t : T}$$

Russell's’ Simple Type Theory

- SST=Th(N,$\otimes$, *Set*)
- Basic types: Numbers
- Constructors: Products, Sets

Theorem *SST is a grammar independent theory*

Higher Order Logic

- STT=Th(N,$\otimes$, Bool,$\Rightarrow$)
- Basic types: Numbers, Bool
- Constructors: Products, Functions

Theorem *HOL is a grammar independent theory*

Frege Structures

$$\frac{a : P \qquad\qquad b : P}{a \wedge b : P} \qquad\qquad \frac{a : P \qquad\qquad b : P}{a \vee b : P}$$

$$\frac{a[x] : P}{\forall x \cdot a : P} \qquad\qquad \frac{a[x] : P}{\exists x \cdot a : P}$$

$$\frac{a : T \qquad\qquad b : T}{a \wedge b : T}$$

Constructive Type Theory

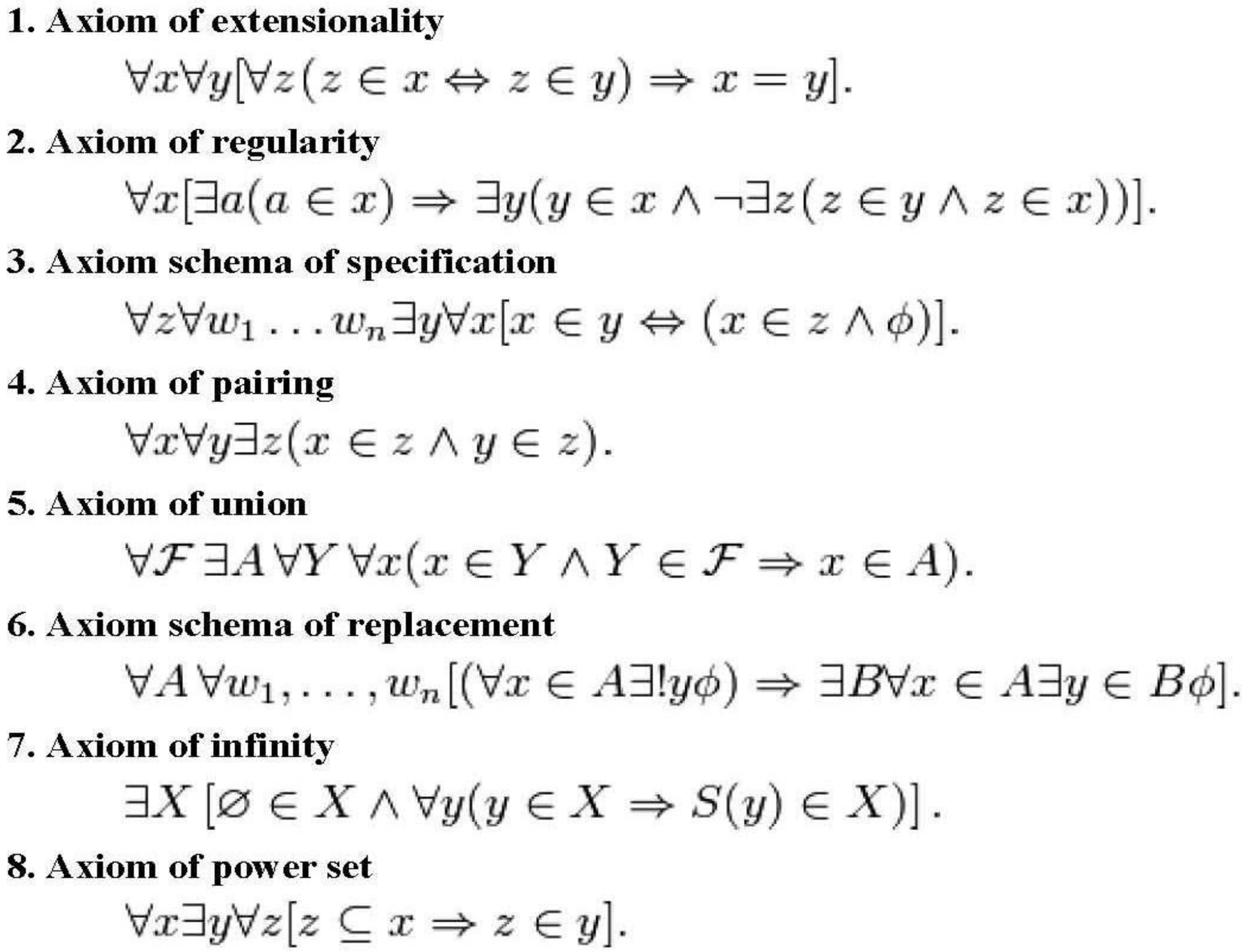
$$\frac{T\ \mathit{type} \qquad S\ \mathit{type}}{T \oplus S\ \mathit{type}}$$

$$\frac{a:T \qquad b:T}{I[T,a,b]\ \mathit{type}}$$

$$\frac{x:T \vDash S\ \ \mathit{type}}{\Sigma x:T.S\ \ \ \mathit{type}}$$

$$\frac{x:T \vDash S\ \ \mathit{type}}{\Pi x:T.S\ \ \ \mathit{type}}$$

ZF



1. Axiom of extensionality

$$\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y].$$

2. Axiom of regularity

$$\forall x [\exists a (a \in x) \Rightarrow \exists y (y \in x \wedge \neg \exists z (z \in y \wedge z \in x))].$$

3. Axiom schema of specification

$$\forall z \forall w_1 \ldots w_n \exists y \forall x [x \in y \Leftrightarrow (x \in z \wedge \phi)].$$

4. Axiom of pairing

$$\forall x \forall y \exists z (x \in z \wedge y \in z).$$

5. Axiom of union

$$\forall \mathcal{F} \, \exists A \, \forall Y \, \forall x (x \in Y \wedge Y \in \mathcal{F} \Rightarrow x \in A).$$

6. Axiom schema of replacement

$$\forall A \, \forall w_1, \ldots, w_n [(\forall x \in A \exists ! y \phi) \Rightarrow \exists B \forall x \in A \exists y \in B \phi].$$

7. Axiom of infinity

$$\exists X \, [\varnothing \in X \wedge \forall y (y \in X \Rightarrow S(y) \in X)] \, .$$

8. Axiom of power set

$$\forall x \exists y \forall z [z \subseteq x \Rightarrow z \in y].$$

The Type of Sets

$$\frac{a : Set \qquad b : Set}{\{a, b\} : Set} \qquad \frac{a : Set}{\cup a : Set}$$

$$\frac{a : Set}{\mathcal{P}a : Set} \qquad N : Set$$

.....

Mathematical Ontology

- Not all notions may be taken as sui generis.
- To formulate a branch of mathematics requires a choice of basic and complex type constructors.
- Different formulations of real numbers.
- ... mathematics is a *motley* of techniques and proofs. Wittgenstein: *Remarks on the Foundations of Mathematics*
- It is also a motley of definitions, theories, structures and systems.

TPL and Foundations

- I see the logician’s function as the articulation and study of the variety of logical systems and structures thrown up by mathematical practice.
- TPL allows a mix and match policy – chose your type constructors and glue them together.

V. DEFINITIONS

Definitional Activity

Definitions introduce new named

1. Relations
2. Operations
3. Objects
4. Types

Logically, this is a *definitional* activity

Definitions

Within TPL such definitions take the following form.

Where

$$x_1{:}T_1,\ldots\ldots,x_n{:}T_n \models \phi[x_1,\ldots,x_n] \text{ prop}$$

We may (conservatively) introduce a new relation

$$R \Bumpeq [x_1{:}T_1,\ldots,x_n{:}T_n\,.\,\phi[x_1,\ldots,x_n]]$$

Axiom

- This is governed by

$$\forall x_1{:}T_1,\ldots,x_n{:}T_n\,.R(x_1,\ldots,x_n) \leftrightarrow \phi[x_1,\ldots,x_n]$$

- *The addition of such a relation results in new theory* TPL_R.
- Applies to any theory of types.

Functions and Definite Descriptions

1. If $\forall x{:}T.\exists ! y{:}S.\phi[x,y]$

Then we may conservatively introduce a new function symbol that satisfies

$$\forall x{:}T.\forall y{:}S.\ \mathrm{F}(x)=y \Leftrightarrow \phi[x,y]$$

2. If $\exists !_{x:T}.\beta[x]$

Then we may introduce a object symbol

$$\mu x.\ \beta[x]$$

Such that $\beta[\mu x.\ \beta[x]] \wedge \forall x{:}T.\ \beta[x] \Rightarrow (x = \mu x.\ \beta[x])$

Admissible Specifications

In the classical theory of definitions, there are coherence/consistency constraints.

Conservative: By defining new things, one cannot deduce anything new about old ones.

Eliminability: Anything said about new things can be reduced to something said about old ones

Constraints in TPL

Conservative: Any prop φ *of TPL*

$TPL_R \models \varphi$ *implies in* $TPL \models \varphi$

Elimination: For any prop φ *of* TPL_R *there is a prop* ψ *of TPL such that* $TPL_R \models \varphi \leftrightarrow \psi$

Group Theory

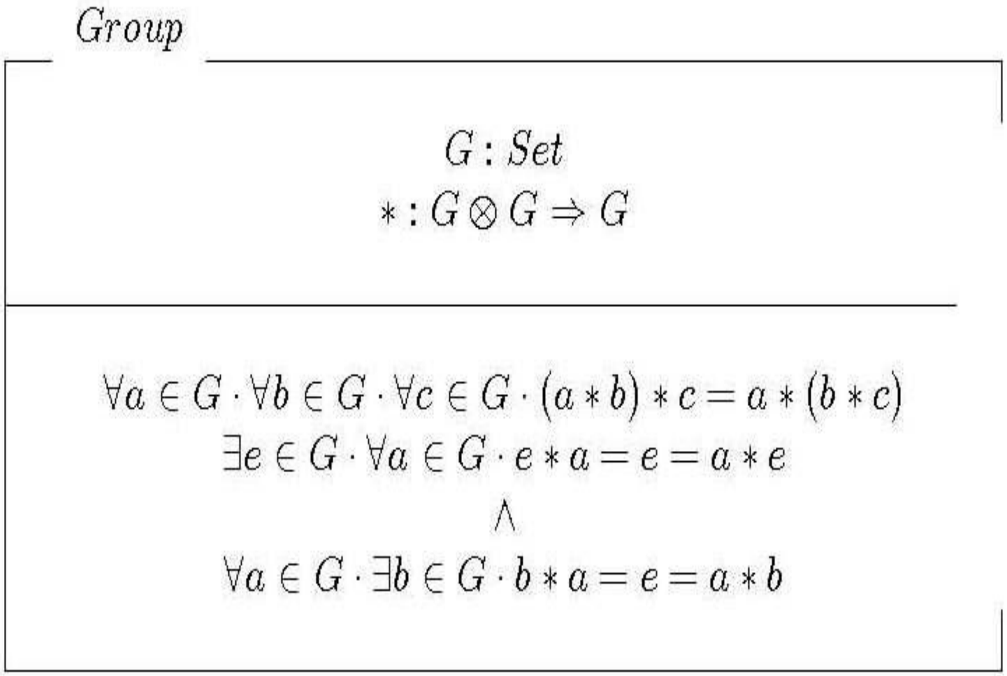
- Let T be a type in GTh=Th(N,$\otimes$, $\Rightarrow$)

$$G \Bumpeq [z{:}\ T \otimes T \Rightarrow T.\ \phi[z]]$$

Where ϕ[*z*] are the axioms of group theory.

The notion of group depends upon the type.

Groups in Set Theory



Group

$$G : Set$$

$$* : G \otimes G \Rightarrow G$$

$$\forall a \in G \cdot \forall b \in G \cdot \forall c \in G \cdot (a * b) * c = a * (b * c)$$

$$\exists e \in G \cdot \forall a \in G \cdot e * a = e = a * e$$

$$\wedge$$

$$\forall a \in G \cdot \exists b \in G \cdot b * a = e = a * b$$

Topological Space in a Theory with Sets

Topological Space

$F: Set(Set(U))$

$\emptyset \in F$

$U \in F$

$G \subseteq F \Rightarrow \cup G \in F$

$f, g \in F \Rightarrow f \cap g \in F$

VI. SUBTYPES

Why Subtypes

- Expressive power
- Enables logical information to be incorporated into type stipulation
- Allows more fine grained types and so more total functions.

Subtypes

Sep₀

$$\frac{x : T \vdash \phi \; prop}{\{x : T \mid \phi\} \;\; type}$$

Sep₁

$$\frac{x : T \vdash \phi \; prop \qquad\qquad a : T \quad \phi[a/x]}{a : \{x : T \mid \phi\}}$$

Sep₂

$$\frac{a : \{x : T \mid \phi\}}{a : T}$$

Sep₃

$$\frac{a : \{x : T \mid \phi\}}{\phi[a/x]}$$

The Addition of Subtypes is Conservative (very often)

Each type proposition and *T* is translated (Via *)
Types break into 2 parts

i. A type T_+
ii. A predicate T_- over T_+

Such that

Theorem: *If* $\Gamma \vDash t{:}T$ *then* $\Gamma^* \vDash t^*{:}T_+$ *and*
$\Gamma^* \vDash T_-(t^*)$

We shall see applications shortly.

VII. ABSTRACT TYPES

Dependent Products

$$\frac{x : T \vdash S[x]\ \mathit{type}}{\sum x : T \cdot S\ \mathit{type}} \qquad \frac{t : T \qquad s : S[t/x]}{(t, s) : \sum x : T \cdot S}$$

$$\frac{s : \sum x : T \cdot S}{s_1 : T} \qquad \frac{s : \sum x : T \cdot S}{s_2 : S[s_1/x]}$$

$$\frac{s : \sum x : T \cdot S}{s =_{\Sigma x : T \cdot S} (s_1, s_2)}$$

Textbook Definition

A group is a set, *G*, together with an operation that combines any two elements *a* and *b* to form another element, denoted

$$a \bullet b.$$

To qualify as a group, the set and operation, $(G, \bullet)$, must satisfy four requirements known as the group axioms.

Groups as Dependent Product types

$$G=\{f:\sum u:\textbf{\textit{Set}}.(u\otimes u\Rightarrow u)\ .\ Groupaxioms(f)\}$$

The elements are pairs consisting of a group and an operation on it which together satisfy the group axioms. This is direct formalisation of the informal notion. It requires subtypes and dependent product types

VIII. POLYMORPHISM IN MATHEMATICS

Explicit Polymorphism in TPL

For each theory Th, add a *universal* type ***U*** *which is closed under the type constructors of the theory e.g.*

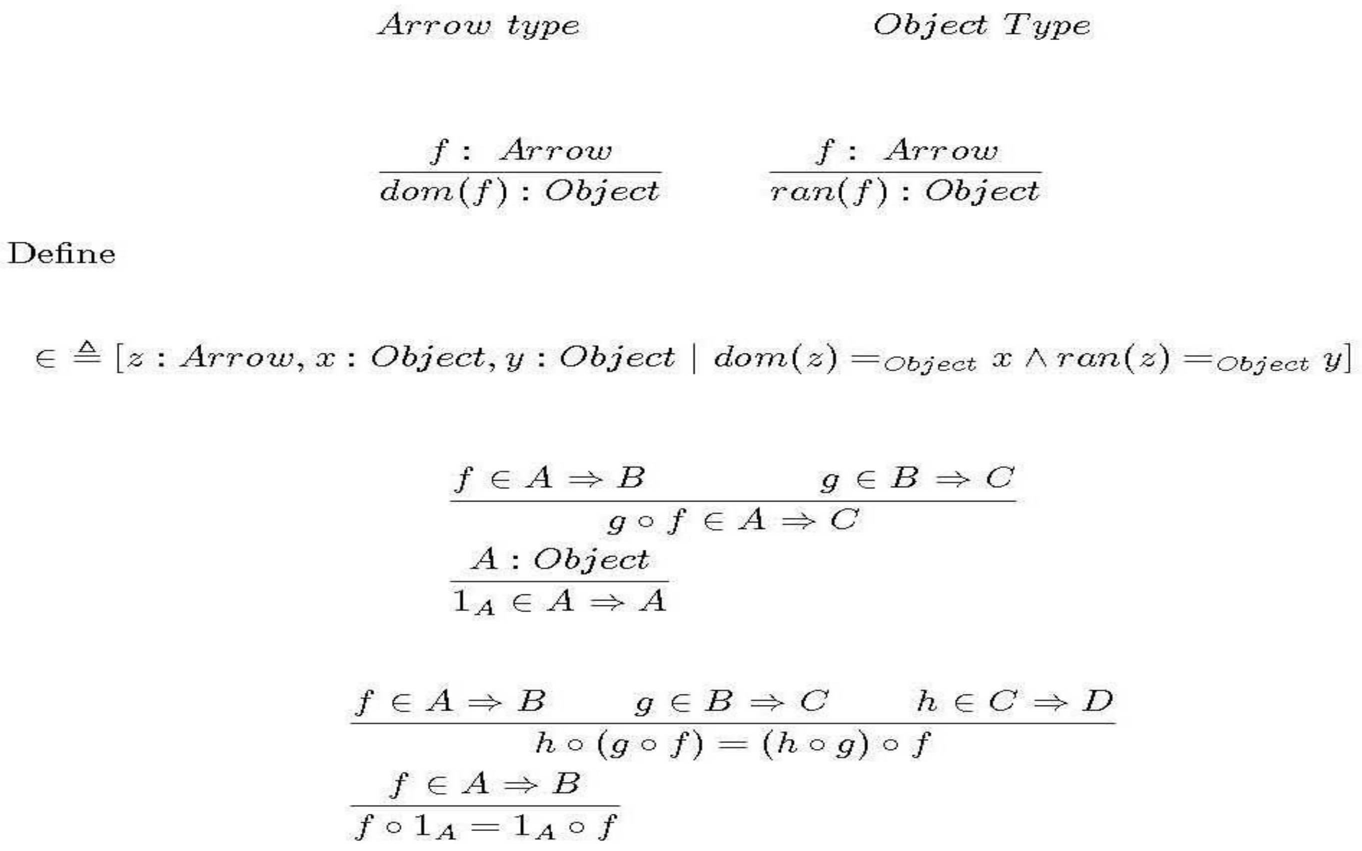
$$\frac{A : \mathrm{U} \qquad B : \mathrm{U}}{A \otimes B : \mathrm{U}} \qquad \frac{A : \mathrm{U} \qquad B : \mathrm{U}}{A \Rightarrow B : \mathrm{U}} \qquad \frac{A : \mathrm{U}}{Set(A) : \mathrm{U}}$$

Plus ***U*** *type*

Categories

- A *category C* consists of the following three mathematical entities:
- ob(*C*), whose elements are called *objects*;
- hom(*C*), whose elements are called *arrows*. Each arrow *f* has a unique *source object a* and *target object b*. We write $f: a \to b$, and we say "*f* is a morphism from *a* to *b*". We write hom(*a*, *b*) (or Hom(*a*, *b*), or $\mathrm{hom}_C(a, b)$, or Mor(*a*, *b*), or *C*(*a*, *b*)) to denote the *hom-class* of all morphisms from *a* to *b*.
- A binary operation, called *composition of arrows*, such that .....

Category Theory



$$\mathit{Arrow\ type} \qquad\qquad \mathit{Object\ Type}$$

$$\frac{f:\ \mathit{Arrow}}{\mathit{dom}(f):\mathit{Object}} \qquad \frac{f:\ \mathit{Arrow}}{\mathit{ran}(f):\mathit{Object}}$$

Define

$$\in\ \triangleq [z:\mathit{Arrow}, x:\mathit{Object}, y:\mathit{Object} \mid \mathit{dom}(z) =_{\mathit{Object}} x \wedge \mathit{ran}(z) =_{\mathit{Object}} y]$$

$$\frac{f \in A \Rightarrow B \qquad\qquad g \in B \Rightarrow C}{g \circ f \in A \Rightarrow C}$$

$$\frac{A:\mathit{Object}}{1_A \in A \Rightarrow A}$$

$$\frac{f \in A \Rightarrow B \qquad g \in B \Rightarrow C \qquad h \in C \Rightarrow D}{h \circ (g \circ f) = (h \circ g) \circ f}$$

$$\frac{f \in A \Rightarrow B}{f \circ 1_A = 1_A \circ f}$$

Categories as Abstract Types

$G=\{f:\sum u:\boldsymbol{U}.(u\otimes u\Rightarrow u)\ .\ \textit{Category axioms}(f)\}$

- *This is a type whose elements are categories but where the category object is a type rather than a set.*
- *Equality is between categories – or rather the base type* $\sum u:U.(u\otimes u\Rightarrow u)\otimes u$.

IX. SORTS AND TYPES

Types as Sorts

- Criteria of counting and identity
- Is this group identical to that topological space?
- Strawson, Geach, Wiggins on sorts in general. How do their views apply to mathematical ontology? What impact do these studies have on the present view?

X. OTHER APPLICATIONS

Logics of Computation

- See CSEE slides on my web page

Application to Formal Ontology

Exercise: Develop Type theories with types such as

Events

Individuals, properties and relations

Time

Mereology

Collections

.

.

Exercise.

Formalise a standard formal ontology such as that of Smith or Zalta. PhD topics.

Application to Semantics

Ontology required for semantics theory

Propositions, events, properties,...

See forthcoming work of Chris Fox