Shape from Radiological Density

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In this paper we propose a strategy to solve the problem of recovering the 3-D shape of anatomical structures from single X-ray images, i.e., the problem of Shape from Radiological Density (SFRD). In order to overcome the noninvertibility of the process of image generation, we formulate a minimal set of physical assumptions that are used to constrain SFRD and to transform it into a well-posed problem. Our shape recovery strategy requires the solution of four problems: (a) linearization of the process of X-ray image generation, (b) image segmentation, (c) estimation of a map of the local thickness of each anatomical structure of interest, and (d) recovery of the 3-D shape of each structure from its boundaries and thickness map. In this paper we assume that problems (a) and (b) have already been faced, and propose a solution for problems (c) and (d). Experimental results on synthetic images, X-ray images of phantoms, and real radiograms are reported.

1. INTRODUCTION

The recovery of the three-dimensional shape of anatomical structures is one of the most important problems in the field of medical imaging as the quantitative, computer-based assessment of such a shape and its changes play an important role in clinical and research studies on a number of diseases.

A frequently studied class of solutions to this problem consists of performing the regional segmentation of a big enough sequence of tomographic (e.g., MR, CT, or echographic) images and stacking the segmented slices (sometimes with interpolation) to obtain volumetric representations of the 3-D shape of the imaged structures [1–6]. Alternatively, detailed surface representations can be obtained from the boundaries of the structures of interest detected in each slice by means of surface fitting or interpolation methods [7–15]. Surface fitting approaches can also be used for the recovery of the approximate shape of an anatomical structure from a small set of tomographic or radiographic images when the boundaries of such a structure or other surface landmarks (obtained, for example, by matching fiducial points in different views) are available [16–19].

The above-mentioned methods use the output data of a segmentation or a feature-matching algorithm as geometric constraints for the recovery of 3-D shape. However, in the case of X-ray projective imaging, an additional, important source of 3-D information is available: the selective absorption of X-ray photons by the different tissues being imaged.

Such a source of information is used, for example, in computed tomography for the reconstruction of a density image from a complete set of projections on the grounds of Radon’s theorem [20]. Unfortunately, when the number of projections available is small, image reconstruction becomes an extremely ill-posed problem that can be (approximately) solved only with the formulation of a set of strong assumptions about the structures in the scene [21–23].

A typical assumption formulated when a very small number (2–4) of projections is available is that only a single structure is present in the scene.\(^1\) The additional hypotheses that the structure has a constant density, that the rest of the imaged volume is empty and/or that the cross sections of the structure respect predefined constraints allow the reconstruction of such a structure. For example, the assumption that most of the imaged volume is empty allows for 3-D reconstruction of vessels from a very small number

\(^1\) It should be noted that, under the hypothesis of a single structure in the scene, the reconstruction of an image representing the density of a structure is actually equivalent to the regional segmentation of a cross section of such a structure and, thus, the reconstruction of several parallel slices is equivalent to instantiating a 3-D volumetric shape representation.
of X-ray images [24]. In [25], the assumptions of elliptical cross section and constant density of vessels allow for the estimation of the shape of coronary arteries from two X-ray projections (biplane angiograms). The hypothesis of elliptical cross section can be removed if other (less restrictive) assumptions on the characteristics of vessels are formulated [26–28]. Convex symmetric cross sections with piecewise linear boundaries are assumed in [29] for heart reconstruction from two X-ray projections. Fewer restrictions but much more a priori knowledge on the expected shape of cross sections are used in [30] for the reconstruction of ventricular shape from biplane angiograms. Regularity of ventricular cross sections with respect to the two projection directions is hypothesized in [31].

In most of the above-mentioned methods the space that does not belong to the structure of interest is assumed to be empty of hypothesizing that such a structure is much denser than the other structures in the scene or that a kind of background subtraction can be performed. However, the first hypothesis is generally not valid for anatomical structures and background subtraction can be performed only for cave structures injected with iodine dye (in such a case, after linearization, a preinjection image can be subtracted from the postinjection one). Actually, most of the methods described in literature have been applied to cardiovascular structures.

The strong assumptions described above prevent the applicability of these methods to most anatomical structures of diagnostic interest. In addition, the requirement of having at least two projections rules out completely the most interesting possibility of recovering the shape of anatomical structures starting from single, conventional radiograms: the cheapest, most widespread kind of medical images.

In this paper we present a method for the recovery of the shape of anatomical structures from X-ray images, a problem that we term shape from radiological density (SFRD). The method presents the following features: (a) only a single X-ray image is required (even if the method is adequate for

FIG. 1. Cylindrical surface model.

FIG. 2. Action of boundary points (a) and thickness data (b).
any number of projections); (b) shape recovery is performed by exploiting both geometric and densitometric constraints; (c) the shape of more than one structure can be recovered, (d) in addition to the structure(s) of interest, background and undesired structures can be present in the image, thus allowing also for shape recovery of a large class of noncave structures; (e) shape recovery is not performed on a slice-by-slice basis but all the input data cooperate in the instantiation of a single 3-D surface model; (f) noisy, incomplete, and inconsistent data are acceptable inputs.

In the form presented in this paper the method is particularly suited to recover the shape of elongated structures, such as cardiovascular structures and most of the bones of the human body. However, as shown in the experiments, it works very well aslo for compact anatomical structures (such as a head or a brain). Extensions to nonelongated structures are also possible.

The method requires that the linearization of the process of image generation and the segmentation of the radiogram have been (approximately) performed. It includes two steps: estimation of the local thickness of each anatomical structure of interest and recovery of the 3-D shape of each structure. The theory behind these steps is described in Section 2, while in Section 3 we provide more details on the practical implementation of such a theory. The experimental results of our method for SFRD are given in Section 4, while in Section 5 we make some final comments.

2. METHOD

From a mathematical point of view, SFRD is an inverse, severely ill-posed problem that has an infinite number of solutions. To transform it into a problem with a single solution, the typical characteristics of the structures to be recovered must be taken into account. To avoid an over-restriction of the solution space, we have considered only the following two assumptions: (A1) the density of each structure is approximately constant and (A2) the surface of each structure is smooth. These assumptions are valid for a large class of anatomical structures and also for many other natural or man-made objects.

As already mentioned, prerequisites for our method for SFRD are linearization and segmentation of the radiogram, even though approximate results, such as the ones obtained by automated procedures on noisy images, are acceptable. From such data, the local thickness and the 3-D shape of each imaged structure are computed as explained in the following two subsections.

2.1. Thickness Map Estimation

The first step of the method is the estimation of an image, termed thickness map, for each anatomical structure of interest, whose gray levels \( s(x, y) \) are proportional to the thickness of such a structure along the ray of projection that crosses the image plane in \((x, y)\).

In the hypothesis of orthographic projection, the process of formation of X-ray projective images is approximately ruled by the equation [32–34]

\[
N_e(x, y) = N_i(x, y) \exp \left( -\int_0^d \mu(x, y, z) \, dz \right),
\]

where \( \mu(x, y, z) \) is the linear absorption coefficient of the volume element of position \((x, y, z)\) of a tissue of thickness \(d\), \(N_i(x, y)\) is the number of X photons entering the tissue in \((x, y, d)\), and \(N_e(x, y)\) is the number of photons that exit such a tissue in \((x, y, 0)\). We define radiological density the expression \( \int_0^d \mu(x, y, z) \, dz \).

The gray levels of the pixels of an X-ray image are
FIG. 4. Estimated thickness maps of the ellipsoid (a) and the cylinder (b) shown in Fig. 3a.

FIG. 5. Recovered shape of the structures shown in Fig. 3a. The cylinder has been cut to show the internal ellipsoid.
The thickness $s(x, y)$ is zero for points $(x, y)$ outside the boundary $\gamma_i$ of structure $i$ (the curves $\gamma_i$'s are provided as input along with the radiological density). Therefore, we can rewrite the previous equation as

$$I(x, y) = \sum_{i=1}^{N} \mu_i s_i(x, y).$$

By applying the gradient operator $\nabla = [\partial/\partial x, \partial/\partial y]$ to both sides of this equation we obtain

$$\nabla I(x, y) = \sum_{i=1}^{N} \mu_i \nabla s_i(x, y).$$

On the grounds of the smoothness assumption A2 we can hypothesize that the largest changes in the thickness of an anatomical structure occur near the apparent contour of such a structure, i.e.,

$$\nabla s_i(x, y) \begin{cases} 
\neq 0 & \text{if } (x, y) \text{ is near } \gamma_i, \\
= 0 & \text{otherwise.}
\end{cases}$$

To corroborate this hypothesis, let us consider the example of a sphere. The thickness of such a structure is given by $s(x, y) = 2\sqrt{r^2 - (x - x_c)^2 - (y - y_c)^2}$, $r$ and $(x_c, y_c)$ being the radius of the sphere and the projection of its center on the image plane, respectively. Simple calculations can show that $s(x, y)$ reaches 25% of its maximum value at a distance of 0.03$r$ from the boundary, 50% at 0.13$r$ and 75% at 0.34$r$.

Now, if we consider that the contours of different structures are usually close to one another only for small tracts near the crossings (if any) of the curves $\gamma_i$, from the previous two equations we obtain

$$\nabla I(x, y) = \begin{cases} 
\mu_i \nabla s_i(x, y) & \text{if } \exists h \text{ such that } (x, y) \text{ is near } \gamma_h, \\
0 & \text{otherwise.}
\end{cases}$$

In other words, near the boundary of an anatomical structure the image gradient is proportional to the gradient of the thickness of such a structure only.

Following [35], if we hypothesize on the basis of A2 that the thickness maps $s_i(x, y)$ are differentiable to the second order with continuous second derivatives, thanks to Schwartz's and Green's theorems we have

$$s_i(x, y) - s_i(x_0, y_0) = \int_{(x_0, y_0)}^{(x, y)} \left( \frac{\partial s_i(x, y)}{\partial x} \, dx + \frac{\partial s_i(x, y)}{\partial y} \, dy \right),$$

independently of the adopted integration path.

If we choose as starting point $(x_0, y_0)$ a point belonging to $\gamma_h$ (the boundary of the $h$th structure), so that $s_h(x_0,$
$y_0 = 0$, and as ending point a generic point $(x, y)$ inside $\gamma_h$, the previous equation transforms into

$$s_h(x, y) = \int_0^1 \nabla s_h(x(t), y(t)) \cdot \mathbf{n}(t) \, dt$$

$$= \frac{1}{\mu_h} \int_0^1 \nabla I(x(t), y(t)) \cdot \mathbf{n}(t) \, dt.$$ (1)

where $[x(t), y(t)]$ is an arbitrary curve such that $[x(0), y(0)] = (x_0, y_0)$ and $[x(I), y(I)] = (x, y)$, and $\mathbf{n}(t) = [\dot{x}(t), \dot{y}(t)]$ is the tangent vector of such a curve. This equation provides a method for estimating the local thickness of a structure from the image gradient.

The coefficient $\mu_h$ in Eq. (1) is unknown. Therefore, in theory, the local thickness of each structure of interest can be recovered up to a scaling factor only. However, in biological tissues, $\mu_h$ usually has a value in the range 0.21–1.05 cm$^{-1}$ [36]. This provides a lower and upper limit

FIG. 7. Estimated thickness maps (before normalization) of the two vials (a) and (b) shown in Fig. 6a.

FIG. 8. Two views (a) and (b) of the shape of the two vials shown in Fig. 6a.
for the aforementioned degree of freedom. Much narrower limits for $\mu_b$ can be assumed if the boundaries provided as input for SFRD are labeled, e.g., as bone, soft tissue, and fat; since, in this case, apart from a small interpatient variability, the linear absorption coefficient is known. Alternatively, the linear absorption coefficient of a structure can be guessed on the basis of the dimensions of its contour or estimated from the boundaries of the same structure in images taken from different viewpoints.

2.2. 3-D Shape Recovery

The data provided by the process of thickness estimation and the boundaries of the structures of interest are often noisy, incomplete, and inconsistent. The phase of shape recovery is aimed at integrating all the information about each structure into a single consistent 3-D model.

Following an approach which is quite common in computer vision (see for example [15, 37–39]), in order to make this problem well-posed, we have used an implementation of physical inspiration for the smoothness assumption A2. The surface has been modeled as an elastic thin-surface $S$ under the action of external forces generated by springs which deform the surface so as to make it best explain the data. Therefore, the problem of shape recovery is solved by mathematically modeling the process of relaxation of the elastic surface toward a state of minimum potential energy and finding the minimum energy surface.

The potential energy $E_{\text{total}}$ of the surface-plus-springs system is the sum of three terms: the elastic energy of the springs which account for boundary points $E_{\text{boundary}}$, the elastic energy of the springs which model the action of thickness data $E_{\text{density}}$, and the internal energy of the surface $E_{\text{surface}}$. In order to express such terms mathematically, a reference system and a mathematical representation for the surface $S$ must first be chosen.

In this work we have adopted a cylindroidal model of $S$, similar to the one used in [38, 40], which can be represented by the parametric equation

$$x(u, v) = ve_1 + f(u, v) e_2 + f(u, v) \sin(u) e_3,$$

where $f(u, v)$ is a twice differentiable function, periodic with respect to $u$, $(u, v) \in D = [0, 2\pi) \times [0, L]$, $L$ being the length of the cylindroidal model, and $e_1, e_2, e_3$ the unit vectors of an object-centered Cartesian reference (see Fig. 10).
1. The unit vector $\mathbf{e}_3$ of such a reference is parallel to the X-ray projection direction (the $z$ axis), $\mathbf{e}_1$ is parallel to the axis of inertia of the contour of the structure being recovered and the origin is aligned with the centroid of such a contour. We have chosen a cylindroidal model as it is particularly suited for representing elongated structures, such as cardiovascular structures and most of the bones of the human body, as well as many compact structures. However, with minor changes, the shape-recovery procedure described below could be used for other surface models such as the one described in [15].

2.2.1. Evaluation of $E_{\text{boundary}}$. The boundary of a given structure in the image plane is the projection of a 3-D curve belonging to the surface of the structure, termed apparent contour. The apparent contour does not necessarily lie on a plane parallel to the image plane; therefore, this fact must be taken into account when modeling the action of boundary points. To avoid over-constraining the surface to be recovered, we have modeled boundary points as springs attached to the apparent contour of the surface and to trolleys that can move along the $z$ axis, as shown in Fig. 2a. (Similar constraints have been used in [38, 40, 41].)

According to this model, if we denote with $(v_i, r_i)$ the coordinates of the $i$th boundary point in the reference system of unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, we can express the elastic energy of the springs which account for boundary points as

$$E_{\text{boundary}}(f) = \frac{1}{2} \sum_{i=1}^{P} \beta_i (f(u_i, v_i) \cos u_i - r_i)^2,$$

where $\beta_i$ is the stiffness of such springs and $u_i$ is such that

$$f(u_i, v_i) \cos u_i = \begin{cases} \max_u f(u, v_i) \cos u & \text{if } r_i > 0, \\ \min_u f(u, v_i) \cos u & \text{otherwise}. \end{cases}$$

2.2.2. Evaluation of $E_{\text{density}}$. The estimated thickness map provides information on the local thickness of the structure whose shape is to be recovered. Therefore, thickness data should be modeled as springs, having a rest-length equal to the estimated thickness, attached to the lower and upper surfaces of the model (see Fig. 2b).

According to this model, if we denoted with $(v(x, y), r(x, y))$ the coordinates of a point $(x, y)$ of the thickness map $s_t(x, y)$, in the reference system of unit vectors $\mathbf{e}_1$ and
$e_2$, we could express the elastic energy of the springs which model the action of thickness data as

$$E_{\text{density}}(f) = \frac{1}{2} \sum_{xy} \beta_d \left[ \sin u_d(x,y) f(u_d(x,y), v(x,y)) + \sin u_d(x,y) f(u_d(x,y), v(x,y)) - s_d(x,y) \right]^2$$

where the summation is performed for all the points $(x, y)$ such that $s_d(x,y) \neq 0$, $\beta_d$ is the stiffness of the springs, and $u_d(x,y)$ and $u_d(x,y)$ are such that

$$f(u_d(x,y), v(x,y)) \cos u_d(x,y) - r(x,y) = \min_{u \in [0, 2\pi]} \left[ f(u, v(x,y)) \cos u - r(x,y) \right]$$

and

$$f(u_d(x,y), v(x,y)) \cos u_d(x,y) - r(x,y) = \min_{u \in [0, 2\pi]} \left[ f(u, v(x,y)) \cos u - r(x,y) \right].$$

This formulation for the densitometric constraints would not overconstrain the surface to be recovered. However, when a single radiogram is used, densitometric constraints of such a form, along with the geometric constraints described above, would leave the 3-D model underconstrained. In addition, this formulation would induce more severe relative errors in the areas of $S$ closest to the apparent contour, that are exactly the parts on which more (and more precise) information is available. Therefore, we have adopted a different formulation in which each thickness constraint (the trolley with springs shown in Fig. 2b) is replaced with two radial springs, one attached to the upper part and the other to the lower part of $S$, which force the surface to be more consistent with thickness data near the boundary.

According to this formulation, we can express the elastic energy of the springs which model the action of thickness data as

$$E_{\text{density}}(f) = \frac{1}{2} \sum_{xy} \beta_d \left[ \left[ f(u_d(x,y), v(x,y)) - s_d(x,y) \right] \frac{1}{\sin u_d(x,y)} \right]^2$$

$$+ \left[ f(u_d(x,y), v(x,y)) - s_d(x,y)(1 - \lambda(x,y)) \right] \frac{1}{\sin u_d(x,y)} \right]^2$$

where $\lambda(x, y)$ is the ratio between the thickness of the upper part of the surface and the total thickness:
2.2.3. Evaluation of $E_{\text{surface}}$. If we imagine the surface $S$ to result from bending a thin plate [42], so that a sort of deformed cylinder is obtained, its potential energy is proportional to the functional

$$E_{\text{surface}}(f) = \int_S \left[ 2H^2 - K \right] dS,$$

(6)

where $H$ and $K$ are the mean and Gaussian curvatures of $S$, respectively. From differential geometry [43] it is known that

$$H = \frac{EN + GL - 2FM}{2 \Delta S^2}, \quad K = \frac{LN - M^2}{\Delta S^2}, \quad dS = \Delta S \, du \, dv,$$

where $E$, $F$, $G$ and $L$, $M$, $N$ are the first and second fundamental coefficients of $S$, respectively, and $\Delta S = |\mathbf{x}_u \times \mathbf{x}_v| = (EG - F^2)^{1/2}$. For our cylindroidal surface straightforward calculations show that

$$E = \mathbf{x}_u \cdot \mathbf{x}_u = f^2 + f_u^2, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v = f_u f_v, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v = 1 + f_v^2,$$

$$L = \mathbf{x}_{uu} \cdot \mathbf{n} = \Delta S^2 + 2f_u^2 - 2f_u f_v, \quad M = \mathbf{x}_{uv} \cdot \mathbf{n} = 2f_{uv} - f_u f_v,$$

$$N = \mathbf{x}_{vv} \cdot \mathbf{n} = \Delta S^2 \left[ \sqrt{f_u^2 + f_v^2 + f_{uv}^2} \right],$$

where subscripts denote the partial derivatives of $\mathbf{x}$ and $f$ with respect to $u$ and/or $v$, and $\mathbf{n} = \mathbf{x}_u \times \mathbf{x}_v / |\mathbf{x}_u \times \mathbf{x}_v|$ is the surface normal. Substitution of these equations into Eq. (6) leads to an express for $E_{\text{surface}}(f)$ of the form

$$E_{\text{surface}}(f) = \int_D \mathcal{J}_{\text{surface}}(f) \, du \, dv,$$

(7)

where $\mathcal{J}_{\text{surface}}(f)$ is a complicated, nonquadratic, differential operator as in [15].

3. IMPLEMENTATION

Some of the details of our implementation of the methods for thickness estimation and shape recovery described

$^2$An alternative expression for the potential energy of $S$ is $E_{\text{surface}} = \frac{1}{2} \int_S \left( k_1^2 + k_2^2 \right) dS$, where $k_1$ and $k_2$ are the principal curvatures of the surface.
in the previous sections are given in the following two subsections.

3.1. **Thickness Map Estimation**

In our implementation we have chosen line segments originating from contour points and having a direction orthogonal to the principal axis of inertia of the contour (the axis of minimum momentum) as integration paths for Eq. (1). The tangent vector of these paths can be expressed as \( \mathbf{n}(t) = [\cos \theta, \sin \theta] \), \( \theta \) being a suitable constant, so that

\[
s_t(x, y) = \frac{1}{\mu_b} \int_0^l \frac{\partial I}{\partial x} \bigg|_{(x_0, y_0)} \cos \theta + \frac{\partial I}{\partial y} \bigg|_{(x_0, y_0)} \sin \theta \, dt.
\]

The estimation of the thickness map of a structure is performed by estimating the partial derivatives of \( I(x, y) \) and numerically integrating the previous equation. Regularized estimates of the partial derivatives of \( I(x, y) \) are obtained by convolving \( I(x, y) \) with the two kernels that result from evaluating the partial derivatives of a two-dimensional Gaussian function with standard deviation

![FIG. 14. Estimated thickness maps of the bone (a) and the soft tissue (b) shown in Fig. 13a.](image)

![FIG. 15. Two views of the finger in Fig. 6a. In (b) the surface of soft tissue has been cut to show the bone surface.](image)
Numerical integration is performed by bilinearly interpolating such estimates at the sampling points of an extended trapezoidal integration formula. Integration is stopped when the axis of inertia or the boundary of the current or another structure is encountered.

It should be noted that this algorithm attempts to estimate the thickness of a structure also in its internal parts. Of course, estimation errors are greater than for peripheral points, but this fact has been accounted for in the method for 3-D shape recovery. Integration is stopped only when the boundary of another object is encountered, since in this case gray level variations have to be attributed to the presence of such an object. In such a situation $s_p(x, y)$ of the first structure is considered to be undefined for points within the contour of the second structure.

### 3.2. 3-D Shape Recovery

In Section 2.2 we found the analytic expressions of the three components of the energy function $E_{total}(f)$. In order to recover the shape of a structure from its boundary and thickness map, the energy functional has to be discretized and then minimized. Unfortunately, as all the components of $E_{total}(f)$ are nonquadratic, direct discretization leads to a nonquadratic discrete functional that is hard to calculate and minimize. Therefore, we have adopted a strategy, similar to the one described in [15], in which first a quadratic functional that approximates $E_{total}$ is derived, and then a discrete form of it is obtained. As the approximation and discretization of the three components of $E_{total}$ (especially $E_{surface}$) require a considerable amount of calculations, for the sake of brevity, in the following subsection we provide only an outline of the steps needed for such operations. Then, we describe strategies for the elimination of the degrees of freedom present in models recovered from a single radiogram.

#### 3.2.1. Approximation and Discretization of $E_{total}$

We first hypothesize that the shape to be recovered, represented by the function $f_{min}$ that minimizes $E_{total}(f)$, can be obtained with small deformations from a known reference configuration, represented by the function $f_0$.

In this case, the nonquadratic differential operator $\mathcal{L}_{surface}(f)$ in Eq. (7) can be approximated with its truncated Taylor expansion $\mathcal{L}_{surface}(\theta, f, f_0)$ about $f_0$ [44]. The resulting approximation for $E_{surface}(f)$ is a quadratic function $E(f)$ that is discretized with the finite element method (FEM) [45]. Basically, FEM consists in dividing the domain $D$ into a set of small rectangular subdomains and hypothesizing that the function $f_{min}$ can be properly represented by quadratic functions in such subdomains. As each of such quadratic functions depends uniquely (and linearly) on the values, termed nodal variables, taken by $f$ in the vertex of the subdomains, the substitution of such functions into $E_{surface}(f)$ leads directly to the required discrete quadratic approximation for $E_{surface}(f)$.

If the values $u_1, u_2(x, y), u_3(x, y)$ and $\lambda(x, y)$ present in the expressions of $E_{boundary}(f)$ and $E_{density}(f)$ were known and constant, such functions would already be quadratic and no approximation would be needed. This suggests that, in the hypothesis of small deformations with respect to the reference configuration represented by $f_0$, we can obtain a quadratic approximation for $E_{boundary}(f)$ and $E_{density}(f)$ by using $f_0$ instead of $f$ in Eqs. (2)-(5). Discretization is then obtained by substituting the values $f(u, v), f(u_1, x, y), f(u_2, x, y), f(u_3, x, y)$ and $f(u, v)$, $f(u_1, x, y)$, $f(u_2, x, y)$, $f(u_3, x, y)$ in the expressions of $E_{boundary}(f)$ and $E_{density}(f)$, with the corresponding nearest nodal variables.

As suggested in [15], the hypothesis of small deformations can be relaxed by using the following procedure. First a reference function $f_0$ of simple form is selected. (In this work we have used a cylinder having an axis parallel to the axis of inertia of the detected boundary and a radius equal to the average distance of the edge points from the axis of inertia.) Then, the function $f_0$ is used to evaluate a first approximation of the energy of the model and to find a first approximation $f_1$ of the function $f_{min}$ that minimizes $E_{total}(f)$. Such an approximation is then used in place of $f_0$ to find a better approximation $f_2$ for $f_{min}$, and the procedure is iterated.

As $f_0$ usually is a rather rough approximation of $f_{min}$, the springs that account for thickness data should not be applied immediately to the model as most of them should be attached to wrong nodes. This could trap the surface model in a local energy minimum in which boundary points are not properly fit. To avoid running this risk, we initially set the spring stiffness $\beta_0 = 0$ and slowly increase its value as better and better approximations for $f_{min}$ become available.

#### 3.2.2. Elimination of Degrees of Freedom

When a single X-ray image is used, the recovery of the shape of anatomical structures is possible only up to a few degrees of freedom. In fact, the position of the models along the projection direction and the shape asymmetries with respect to a plane orthogonal to such a direction cannot be estimated. In addition, as already mentioned in Section 3.1, if the linear absorption coefficient is not known a priori, the local thickness of a structure can be recovered up to a scaling factor only. In our experiments (if not otherwise stated) we have resolved some of these ambiguities by arbitrarily setting the $z$ position of the center of mass of the model and estimating the linear absorption coefficient heuristically.

Asymmetry-related degrees of freedom do not cause any problems to the recovery procedure. The reason is that in the absence of explicit constraints the elasticity of
FIG. 16. Two views of the bones of the hand shown in Figure 12.
Comparison between the Exact Data and the Data Obtained by SFRD Applied to Synthetic X-Ray Projections of an Ellipsoid in the Presence of Uniform Noise of Different Intensities

<table>
<thead>
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<th>Projection along y</th>
<th>Recovered model Cross-sectional difference (%)</th>
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<td>2.5</td>
<td>1.8</td>
<td>6.5</td>
</tr>
</tbody>
</table>

As roughly speaking $\sum_{i=1}^{n} \beta_i$ and $\sum_{j=1}^{m} \beta_j$ play the role of regularization parameters, they should be chosen on the basis of the characteristics of the noise expected to affect the data [46]. For the experiments described in this section we have adopted as default values $\sum_{i=1}^{n} \beta_i = \sum_{j=1}^{m} \beta_j = 10000$; smaller values have been used only in the presence of real X-ray images of very poor quality.

Figure 3a shows a synthetic $128 \times 128$ X-ray image of a cylinder surrounding an ellipsoid whose shapes and linear absorption coefficients are known. The image was generated under the hypothesis that the process of image formation is linear and the projection is orthographic. The boundaries of the cylinder and ellipsoid, detected as the zero crossings of the convolution of the original image with a Laplacian-of-Gaussian mask, are reported in Figs. 3b and 3c.

From these data, the algorithm described in Section 3.1 has estimated two thickness maps shown in Figs. 4a and 4b. Despite the inaccuracy in the computation of the axes of inertia, the approximation in the calculation of the partial derivatives of the image, the accumulation of integration errors and the inexact location of some edge points, both maps are nearly correct. Of course, the map of the cylinder includes an area (the elliptic hole) in which thickness cannot be estimated.

The models recovered from the boundary points in Figs. 3b and 3c and the thickness maps in Figs. 4a and 4b are shown in Fig. 5. The $24 \times 24$ grid adopted for FEM discretization has been superimposed on the surfaces and the cylinder has been cut to show the internal ellipsoid. The same discretization grid has been used also for the other experiments, if not otherwise stated. On a DEC Alpha 300MHz workstation, the computation needed to reconstruct such models varies between 4 and 5s of CPU time depending on the number of constraints available.

Figure 6 shows a $128 \times 128$ X-ray image of two vials...
FIG. 17. Two orthogonal projections of the 3-D CT scan of a head.
containing iodine dyes of known densities (a reference metallic object is also present) and the boundaries of the two vials detected by an automatic segmentation system [47]. The boundaries are incorrect where the two vials overlap and near the ends.

Figures 7a and 7b show the thickness maps recovered from the image in Fig. 6a before normalization by the factors 1/μ₄. The maps are partially affected by the errors in the position of the boundary points and by the low resolution of the original image. Low resolution is responsible for the asymmetry of the thickness estimates with respect to the axis of inertia of the denser vial.

Boundary and thickness data of the vials were integrated by the models shown in Figs. 8a and 8b. In Fig. 8a the viewing direction is parallel to the projection axis; in Fig. 8b the models have been rotated by 45°.

Figures 9a and 9b show a 128 × 128 contrastographic X-ray image of the left ventricle of the heart and the boundaries of the ventricle detected by an automatic segmentation system [47]. Due to the presence of noise, ribs, disuniform distribution of the iodine dye inside the chamber, and nonlinearities, the detected boundaries are incorrect in several places. As a consequence, the estimated thickness map shown in Fig. 10 is rather noisy and inaccurate, especially near the axis of inertia (the linear absorption coefficient of the ventricle has been guessed on the basis of the average distance of the boundary points from such an axis).

Despite such inaccuracies, as illustrated in Fig. 11, the integration of the thickness map with boundary data provides a rather reasonable shape for the ventricle. In addition to the discretization grid, in the figure we have drawn (as white line segments) the springs that model the effect of boundary data.

Figure 12 shows an 853 × 640 radiogram of a hand that has been used for two experiments. In the first experiment we used the 128 × 128 subimage shown in Fig. 13a and the inexact boundaries (Figs. 13b and 13c) extracted by an automatic segmentation algorithm [47] as input data for SFRD.

Despite the incorrect segmentation (fusion) of the three bones due to the low resolution (insufficient to represent the very small inter-bone gaps), the thickness maps of bone and soft tissue seem quite accurate. The recovered 16 × 24 3-D models are shown in Figs. 15a and 15b. Of course, being present in the input boundaries, fusion is also present in the recovered bone surface.

In the second experiment the boundaries of the bones of the image in Fig. 12 have been hand-segmented and sequentially provided as input for SFRD. The resulting 3-D models have been then simply collected to form a model of the skeleton of the hand. Fig. 16 reports two views of the model.

In order to assess the robustness of the two steps of our method for SFRD and to check the effects of the use of multiple views, we have performed some experiments involving the recovery of the thickness map and the 3-D shape of a synthetic ellipsoid starting from one or two orthogonal projections affected by increasing amounts of random noise. The semi-axes of the ellipsoid were oriented along the x, y, and z axes of the reference system; their lengths were 80, 100, and 60 pixels, respectively. Uniform noise with zero mean and appropriate standard deviation was added to the projections of the ellipsoid so as to get the required signal-to-noise ratio (SNR). Statistics were collected by performing ten experiments for each SNR. The results of these experiments are summarized in Table 1.
FIG. 19. Two views of the head shown in Fig. 17.
FIG. 20. Three-dimensional rendering of the 3-D CT scan of the head in Fig. 17.
The second and third columns of the table illustrate the behavior of the method for the estimation of thickness maps. Namely, they report the root mean squared (RMS) error between the exact thickness map and the recovered one (normalized with respect to the maximum thickness) for two different projections of the ellipsoid. It should be noted that although our integration algorithm seems to be slightly biased, it is quite robust with respect to noise (the degradation of the thickness map is relatively small even at the lowest SNR).

Columns 4–7 illustrate the behavior of the 3-D shape recovery method when only a single projection (columns 4 and 5) or two orthogonal projections (columns 6 and 7) are used. In particular, the columns report the difference between the exact and the recovered cross sections of the ellipsoid. Also in this case, the recovery process, although slightly biased, seems nearly insensitive to the image noise with the only exception of the cases in which SNR is extremely poor. The bias is due to the inaccuracies of the thickness map and to the tendency of the cylindroidal model to relax toward a nonelliptical (maximally flat) shape. (Increasing the elasticity of the springs, modeling densitometric constraints would reduce the bias, but would also increase the sensitivity to image noise.) It should also be noted how the exploitation of two orthogonal projections leads to a dramatic reduction of recovery errors.

We have also checked the effectiveness of the method in the presence of multiple projections by using two orthogonal X-ray images of a head obtained by projecting a real 3-D CT scan (including 113 256 × 256 CT slices) which is part of the Chapel Hill Volume Rendering Test Dataset (SoftLab Software Systems Laboratory, University of North Carolina). Figure 17 shows the resulting X-ray images. The plots of the corresponding thickness maps estimated by our integration algorithm are shown in Fig. 18. It should be noted that, due to the presence of the skull, only 21% of the thickness maps of the head could be estimated.

Figure 19 shows two views of the model recovered from this data. Given the relative quality of the simulated projections, in this experiment we have adopted \( \sum \beta_i = 30000 \) and a 60 × 40 discretization grid. The good accuracy of the model can be assessed by comparing it with the images in Fig. 20 obtained by interpolating, segmenting, and rendering the slices included in the original 3-D CT scan. The comparison shows that, although some fine shape details are lost, our SFRD method is able to recover the global shape and all the most important features of a head from the two projections despite the severe incompleteness of the related thickness maps.

5. Final Remarks

In this paper we have presented a method for solving the problem of shape from radiological density along with its implementation and experimentation. The method is based on only two physical assumptions (constant density and smoothness) that are true for many anatomical structures. Under these assumptions a mathematical formulation has been derived which allows for the recovery of the local thickness of each structure of interest. By means of a cylindroidal model of elastic surface, such information, integrated with the geometric constraints imposed by the boundaries detected in the image, has allowed for reliable 3-D shape recovery, despite the inaccuracies in segmentation and linearization of the input data.

When single X-ray images were available, the recovery of the shape of the imaged anatomical structures via SFRD has been possible up to the degrees of freedom described in Section 3.2.2. However, it should be noted that shapes recovered from synthetic images and radiograms of reference objects differ only slightly from the original structures and that the shapes of anatomical structures, although approximate, have been judged to be very realistic by expert physicians.

The robustness of the method increases dramatically when more than one projection of the same structure is available. Thanks to this property, the recovery of complex structures becomes possible even when incomplete thickness maps can only be obtained.

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References


24. G. M. Stiel, L. S. G. Stiel, E. Koltz, and C. A. Nienaber, Digital flashing tomosynthesis: A promising technique for angiocardio-


